A New Canonical Form for Systems of Partial Differential Equations.

By L. B. Robinson.

Introduction.

Some years ago Delassus obtained a canonical form useful for the study of systems of partial differential equations.* It is possible to reduce to this form a very large class of differential systems, but that the form is not absolutely general was discovered almost simultaneously by the author† and by M. Gunter.‡ Were the integration of the given differential system the only question of interest there would be no need to improve the canonical form given by Delassus, for when the given differential system is of the first order the above-mentioned canonical form is always valid, and it has been shown by Riquier that it is always possible to reduce a differential system to the first order.§

But since canonical forms are often useful in the study of the comitants of either differential or algebraic systems, it seems desirable to give a canonical form which has no exceptional cases and will apply equally to a differential system or a system of polynomials. In the following paper the author will construct such a canonical form. It will be noted that the symbols $X_1, X_2 \ldots.$ which are used to represent differential operators could likewise be interpreted as the variables of a system of polynomials.‖. As the new canonical form is closely analogous to that of Delassus, it seems that it will be equally useful in studying the invariants or the characteristics of differential systems.¶ In fact this turns out to be the case in some simple cases considered by the author.

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* *Annales de l'école normale supérieure*, Vol. XIII, 3d Ser., p. 421.

† "Notes from the Mathematical Seminary," J. H. U., 1913. *Comptes Rendus*, 15 Juillet 1913.

‡ *Comptes Rendus*, 14 Avril 1913.

§ Riquier, "Sur les systèmes d'équations aux dérivées partielles," *Annales de l'école normale*, Vol. X, 3d Ser., p. 359.

‖ Delassus, *Annales de l'école normale supérieure*, Vol. XIV, 3d Ser., p. 21.

¶ For further bibliography consult Gunter, "On the Theory of the Characteristics of Systems of Partial Differential Equations" (Russian), *Comptes Rendus*, 13 Octobre 1913; 23 Mars 1914; 20 Avril 1914. Janet, *Comptes Rendus*, 13 Janvier 1913; 27 Octobre 1913. In the three above-mentioned notes of M. Gunter in the *Comptes Rendus*, results somewhat similar to those of the author seem to have been obtained.

§ 1. *A Property of Differential Systems with Regular Initial Conditions.*

Consider a system S of partial differential equations with independent variables $x_1, x_2, \ldots, x_q$ and unknown functions $u_1, u_2, \ldots, u_s$. The equations are supposed to be solved for different derivatives of the u's in such a way that no second member can contain a derivative of order higher than that of the corresponding first member. A derivative which is a first member of S, or a derivative obtained from a first member by differentiation is called a principal derivative. All others are named parametric.* Let the values of the unknowns u and of a suitably chosen number of their parametric derivatives be given as functions of some of the variables x for certain initial values of the remaining variables x. We shall suppose that these initial conditions are all regular in the sense of Riquier.† Let an integer or 'cote' be associated with each of the variables x and the unknowns u in some convenient way,‡ the 'cotes' of the x's being unity, and let the 'cote' of a derivative of an unknown u_p be obtained by adding the order of the derivative to the 'cote' of u_p.

Let Γ be the maximum 'cote' of the first members of the initial conditions. We shall now consider the group of equations $S_{\Gamma+1}$ obtained from the system S, prolonged by repeated differentiations, by selecting those equations in which the 'cote' of the first members is $\Gamma+1$. To simplify writing we shall consider derivatives with respect to four independent variables only as the reasoning would be the same in the general case, and shall use the following abbreviation for a derivative of a typical variable u

$$\frac{\partial^{n_1+n_2+n_3+n_4}u}{\partial x_1^{n_1}\partial x_2^{n_2}\partial x_3^{n_3}\partial x_4^{n_4}}=(n_1, n_2, n_3, n_4).$$

We shall now examine the three derivatives

$$\begin{aligned}\alpha &= (n_1, n_2, n_3, n_4),\\ \beta &= (n_1, n_2, n_3+1, n_4),\\ \gamma &= (n_1, n_2, n_3+1, n_4-1).\end{aligned}$$

Suppose that the first of these is a principal derivative occurring in the group $S_{\Gamma+1}$. The second is obviously obtained from the first by differentiation. Hence, if the first is principal, so is the second. It is to be shown, assuming as we have done, that the initial conditions are regular, that γ is also principal.

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* Riquier, "Les systèmes d'équations aux derivées partielles, Chapter VI.

† *Loc. cit.*, Chapter XII.

‡ Riquier, Chapter VII.

For suppose γ to be parametric. Then it is derived from some parametric derivative of 'cote' Γ, obtained by subtracting unity from the exponent of one of its independent variables. Hence there are four possible primitives of γ, viz.:

$$(n_1, n_2, n_3+1, n_4-2) \quad =\phi_1(x_2, x_3, x_4, x_5 \dots), \tag{1}$$

$$(n_1, n_2, n_3, n_4-1) \quad =\phi_2(x_2, x_3, x_4, x_5 \dots), \tag{2}$$

$$(n_1, n_2-1, n_3+1, n_4-1) =\phi_3(x_2, x_3, x_4, x_5 \dots), \tag{3}$$

$$(n_1-1, n_2, n_3+1, n_4-1) =\phi_4(x_2, x_3, x_4, x_5 \dots). \tag{4}$$

We have written the above initial conditions in the most general form. For in the ϕ's one of the variables, to fix ideas say x, can not occur as an argument because if it did no derivatives of u could be principal, therefore u would be a parameter just as in the following system:

$$\frac{\partial v}{\partial x}=\phi(x, y, u), \quad \frac{\partial w}{\partial x}=\psi(x, y, u).$$

In the above system u is an arbitrary unknown. Let us write $u=\theta(x, y)$ where θ is an arbitrary function of x and y. The derivatives of u are obtained by differentiating θ. They are evidently parametric.

As for $x_2, x_3, \dots,$ we do not assert that they all occur actually in the ϕ's, but when we say the system has regular initial conditions we mean that if x_i occurs actually in one of the functions ϕ then $x_{i+1}, x_{i+2}, \dots, x_q$ also occur actually in the same ϕ.

Now returning to the equations (1), (2), (3), (4) we observe first of all that x_4 can not occur actually in (1), for if it did β could not be a principal derivative. For the same reason x_3 can not appear actually in (2) nor x_2 actually in (3). And, therefore, from the hypotheses that the initial conditions are regular, it follows that x_2 and x_3 do not occur in (1), and x_2 does not occur in (2).

So then we see that we can not obtain γ from the group of initial conditions under the hypothesis that they have the regular form. Therefore γ must be a principal derivative. In exactly the same way we can show that the two derivatives

$$(n_1+1, n_2, n_3, n_4-1),$$
$$(n_1, n_2+1, n_3, n_4-1),$$

must be principal also. We may now announce the following result:

Suppose given a system S where the initial conditions are regular. Let Γ be the maximum 'cote' of the first members of the initial conditions. Let

$$(n_1, n_2, \ldots, n_p)$$

be a typical first member of the group $S_{\Gamma+1}$. Then all derivatives obtained by differentiating the above expression once, with respect to one of the independent variables $x_1, x_2, \ldots, x_p$ which are engaged therein, and then reducing by unity the exponent of the last independent variable involved, are also principal.

Call this property "*K*." As a simple example we may suppose $\frac{\partial^2 u}{\partial x_2^2}$ to be a first member of 'cote' $\Gamma+1$ belonging to a system S involving only two independent variables x_1 and x_2. Differentiate with respect to both of these independent variables. We obtain the derivatives

$$\frac{\partial^3 u}{\partial x_1 \partial x_2^2}, \quad \frac{\partial^3 u}{\partial x_2^3}.$$

Then, if the initial conditions are regular, the group of first members of $S_{\Gamma+1}$ must also contain $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ as well as $\frac{\partial^2 u}{\partial x_2^2}$.

§ 2. *Proof that Systems with the Property K Are Regular.*

Firstly, we shall show that if the given system has the property K, the prolonged system has this property also. For, let

$$(n_1, n_2, \ldots, n_q)$$

be a derivative belonging to the group of first members of the system non-prolonged. Differentiate an arbitrary number of times with respect to each variable and we obtain

$$(n_1+\lambda_1, n_2+\lambda_2, \ldots, n_q+\lambda_q).$$

Differentiate this last derivative with respect to x_l and integrate once with respect to x_q. We obtain

$$(n_1+\lambda_1, n_2+\lambda_2, \ldots, n_l+\lambda_l+1, \ldots, n_q+\lambda_q-1),$$

and this last expression can clearly be derived from

$$(n_1, n_2, \ldots, n_l+1, \ldots, n_q-1),$$

while this last derivative surely exists among the group of first members if we assume that the system has the property K.

Select one of the unknowns involved, for example u, and arrange all its derivatives whatsoever whose 'cote' is Γ in the order* adopted by Delassus, and call this set of derivatives (A). From now on, following the example of Delassus, we shall use the notation

$$X_1^{a_1}, X_2^{a_2}, X_3^{a_3}, \ldots, X_p^{a_p} \equiv \frac{\partial^{a_1+a_2+a_3, \ldots, a_p} u}{\partial x_1^{a_1}, \partial x_2^{a_2}, \partial x_3^{a_3}, \ldots, \partial x_p^{a_p}}.$$

Let $X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}$, which we shall call (a), be a member of the group (A). Differentiate with respect to x_1 and we obtain

$$X_1^{n_1+1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}.$$

Among the group of the derivatives of u there are a certain number from which we can obtain the last one by a single differentiation. One of these is

$$X_1^{n_1+1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p-1},$$

which is anterior to (a) in the set (A). Differentiate (a) with respect to x_2 and we obtain

$$X_1^{n_1}, X_2^{n_2+1}, X_3^{n_3}, \ldots, X_p^{n_p},$$

which can be obtained from

$$X_1^{n_1}, X_2^{n_2+1}, X_3^{n_3}, \ldots, X_p^{n_p-1},$$

which belongs to the set (A). And continuing thus we can show that any derivative of (a) taken once with respect to $x_1, x_2, \ldots, x_{p-1}$ can be obtained by differentiating some member anterior to (a) with respect to x_p if $n_p \geqq 1$, which is the case by hypothesis.

Now differentiate twice with respect to x_1 and we obtain

$$X_1^{n_1+2}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{n_p}.$$

If $n_p > 1$ this can be obtained from $X_1^{n_1+1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}-2}$ by differentiating twice with respect to x_p. If $n_p = 1$ it can be obtained from

$$X_1^{n_1+2}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}-1}$$

by differentiating with respect to x_{p-1}, x_p. This course of reasoning is general and we can say:

* Delassus, "Extension du Théorème de Cauchy aux systemès les plus generaux des équations aux dérivées partielles," *Annales de l'école normale supérieure* (1896), Vol. IX.

Let (A) *be the totality of the derivatives of* u *with respect to* $x_1, x_2, \ldots, x_q$ *which have the 'cote'* Γ. *Let*

$$(a) = X_1^{n_1}, \ldots, X_p^{n_p}, \qquad (n_p \overline{\gtrless} 1)$$

be a member of the set. If we differentiate (a) *once with respect to one of the variables* $x_p, x_{p+1}, \ldots, x_q$, *and perform similar differentiations upon the other elements of* (A), *the totality of derivatives found includes all the derivatives of* u *of 'cote'* $\Gamma+1$. *By differentiating similarly* K *times, a set is formed which includes all derivatives of 'cote'* $\Gamma+K$.

Suppose now that our system has the property K, and consider again a derivative of the set (A):

$$(a) = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}.$$

If it is principal, all its derivatives are principal. If it is parametric, we make the following argument:

Differentiate with respect to $x_p, x_{p+1}, \ldots, x_q$. We obtain

$$\left.\begin{array}{l} X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p+1}, \\ X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}, X_{p+1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}, X_{p+j}, \end{array}\right\} \qquad (\mathrm{I})$$

$$\left.\begin{array}{l} X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}, X_{p+j+1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}, X_q. \end{array}\right\} \qquad (\mathrm{II})$$

If any one of the derivatives belonging to (I) or (II) is principal, then any preceding one is so, on account of the property K, which is assumed to hold good for our system. Hence there are three possibilities:

(α) All derivatives obtained from (a) by a single differentiation with respect to $x_p, x_{p+1}, \ldots, x_q$ are principal.

(β) They are all parametric.

(γ) They are divisible into sets (I) and (II); of which the first is composed of principal derivatives, the second of parametric.

Since (I) is composed of principal derivatives, any expression obtained from it by differentiation is also principal. As for the derivatives of (II), any one obtained by operating with $x_{p+j+1}, x_{p+j+2}, \ldots, x_q$ only may or may not be principal. But if we differentiate with respect to any members of the set $x_1, x_2, \ldots, x_{p+j}$ only, once or a greater number of times, we obtain a derivative which is principal because we could obtain it from (I) also.

We shall try to satisfy the requirements by initial conditions of the fol-

lowing types. If a certain group of first members extracted from our systems satisfies condition (α), we shall write down as initial condition corresponding to this type

$$(a) = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p} = a \text{ constant.}$$

If condition (β) is satisfied, we shall write down as corresponding initial condition,

$$(a) = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p} = \phi(x_p, x_{p+1}, \ldots, x_q).$$

Finally, if condition (γ) is satisfied, we shall write the initial condition thus:

$$(a) = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p} = \phi(x_{p+j+1}, x_{p+j+2}, \ldots, x_q).$$

Note that $x_1, x_2, \ldots, x_{p-1}$ are excluded from ϕ in all three cases. In fact we agree that this exclusion shall make part of our hypotheses.

Having assumed that the above types of initial conditions probably satisfy requirements, we shall now proceed to demonstrate rigorously that our supposition is correct.

We proceed as follows confining our attention to the case (γ), because as a result of the exclusion of $x_1, x_2, \ldots, x_{p-1}$ from the ϕ, the argument would not be altered if conditions (α) or (β) were satisfied.

If we differentiate (a) we shall obtain the initial conditions corresponding to (II) and its derivatives with respect to $x_{p+j+1}, x_{p+j+2}, \ldots, x_q$. The right-hand members are

$$\frac{\partial\phi}{\partial x_{p+j+1}}, \frac{\partial\phi}{\partial x_{p+j+2}}, \ldots, \frac{\partial\phi}{\partial x_q},$$

and their derivatives with respect to the variables involved effectively. Hence it is clear that if ϕ contains actually all the variables in the range

$$x_{p+j+1}, x_{p+j+2}, \ldots, x_q,$$

we should certainly obtain by prolongation all the initial conditions corresponding to the parametric derivatives of 'cote' higher than Γ. The only questions which can now arise are the following:

(1) Can any derivative of the ϕ's correspond to a principal derivative? For in that case the ϕ's could not possibly contain all the variables of the range just mentioned actually.

(2) Can we, by prolonging the set of initial conditions constructed in the manner described in the preceding paragraphs,* obtain two different expressions for the same parametric derivative?

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* The description of the construction of the initial conditions runs from the top of p. 100 to the bottom of p. 101.

If we can answer these two questions in the negative, we shall have demonstrated that the initial conditions constructed in the way just described, and which are obviously regular, correspond exactly to the type of system under consideration, *i. e.*, systems with the property K.

Solution of the First Question.

Firstly, we agree that from now on systems which have the property K shall be called canonical. Then let

$$\alpha = X_1^{n_1},\ X_2^{n_2},\ X_3^{n_3},\ \ldots,\ X_{p-1}^{n_{p-1}},\ X_p^s$$

be a derivative whose 'cote' is Γ, where

$$n_1+n_2+n_3+\ldots+n_{p-1}+S=n-1.$$

Furthermore, let

$$\beta = X_1^{m_1},\ X_2^{m_2},\ X_3^{m_3},\ \ldots,\ X_{p-1}^{m_{p-1}},\ X_p^{s+i}$$

be a derivative whose 'cote' is $\Gamma+1$, where

$$m_1+m_2+m_3+\ldots+m_{p-1}+S+i=n,$$

and where i is an integer positive or negative.

Suppose that the first of these two quantities is parametric. In fact, were it not so, it would not figure among the group of derivatives corresponding to arbitrary initial conditions, and we should not be obliged to consider it at all. Our object is to prove that if we differentiate the expression

$$X_1^{n_1},\ X_2^{n_2},\ X_3^{n_3},\ \ldots,\ X_{p-1}^{n_{p-1}},\ X_p^s = \phi(x_p,\ x_{p+1},\ \ldots,\ x_q)$$

with respect to $x_p,\ x_{p+1},\ \ldots,\ x_q$, we shall not obtain any derivative identical with β or a derivative of β.*

To accomplish this we proceed as follows: Firstly, suppose one of the m's to be greater than the n with the same subscript. We then write side by side

$$(\alpha)\quad X_1^{n_1},\ X_2^{n_2},\ X_3^{n_3},\ \ldots,\ X_{p-1}^{n_{p-1}},\ X_p^s = \phi(x_p,\ x_{p+1},\ \ldots),$$
$$(\beta)\quad X_1^{m_1},\ X_2^{m_2},\ X_3^{m_3},\ \ldots,\ X_{p-1}^{m_{p-1}},\ X_p^{s+i} = M.$$

If we differentiate the first of these with respect to $x_p,\ x_{p+1},\ \ldots,\ x_q$ only, the exponents of $X_1,\ X_2,\ X_3,\ \ldots,\ X_{p-1}$ will not be increased, and therefore one of the m's will always be superior to the n with the same subscript. Therefore,

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* It may happen that $x_p,\ x_{p+1},\ \ldots,\ x_{p+j}$ do not occur actually in ϕ. Then we differentiate with respect to $x_{p+j+1},\ x_{p+j+2},\ \ldots,\ x_q$.

in this case, a derivative of (α) with respect to the specified variables, can not be identical with β or a derivative of β. But if

$$n_r \geqq m_r, \qquad (r=1, 2, \ldots, p-1)$$

for all values of r, we differentiate β as many times as is necessary to make the exponent of X_1 equal to n_1, the exponent of X_2 equal to n_2, and in general the exponent of X_r equal to n_r, where $r=1, 2, \ldots, p-1$. Since $i \geq 1$, we may now differentiate (α) with respect to x_p until s becomes equal to $s+i$. Now we can write the two expressions derived from α and β,

$$\alpha' = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s+i} = \phi_{x_p}^{(i)}(x_p, x_{p+1}, \ldots),$$
$$\beta' = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s+i} = M^{(i-1)}.$$

Now if ϕ contains x_p actually, α' is parametric, while β' is principal, since it is derived from β which is principal. But $\alpha' = \beta'$, therefore if $\phi_{x_p}^{(i)}$ contains x_p actually, we are led to a contradiction. But we shall now show that x_p can not occur actually in ϕ which is the primitive function from which $\phi_{x_p}^{(i)}$ is derived, if the system has the property K.

For we have assumed that

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-1}^{m_{p-1}}, X_p^{s+i}$$

is principal. Therefore,

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}+1}, X_p^{s+i}$$

is also principal. Then from the property K it follows that

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}+1}, X_p^{s+i-1}$$

is principal. A repeated application of the same rule shows us that

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}+2}, X_p^{s+i-2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-2}^{n_{p-2}}, X_{p-1}^{n_{p-1}}, X_p^{s+1},$$

are all principal. Now let us consider the group derived from

$$\alpha = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s} = \phi(x_p, x_{p+1}, \ldots),$$

by differentiating it once with respect to $x_p, x_{p+1}, \ldots, x_q$. We write it

$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s+1},$$
$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s}, X_{p+1},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s}, X_q.$$

The first of these we have just shown to be principal, but in any case it follows from the fact that the first member of the above group is principal that we

must equate α to $\phi(x_{p+1}, x_{p+2}, \ldots, x_q)$, where ϕ may not contain actually all the arguments written down, but surely contains all those posterior to the first one which occurs actually. Then, if we differentiate α with respect to $x_{p+1}, x_{p+2}, \ldots, x_q$, we shall not obtain any derivative identical with β. For we have assumed in this case that $n_i \geqq m_i$ for all values of i. Also the order of β is greater than the order of α by unity. Hence, the exponent of X_p in β is greater than the corresponding exponent in α. And if we prolong α by differentiating with respect to $x_{p+1}, x_{p+2}, \ldots, x_q$, we shall not increase the exponent of X_p. Therefore by that operation we can never obtain a derivative identical with β or with a derivative of β.

The above is a special case given to illustrate the method employed. Let us pass to the consideration of the most general case. Let us compare

$$(\gamma)\ X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-2}^{n_{p-2}}, X_{p-1}^{n_{p-1}}, X_p^{s},$$
$$(\delta)\ X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}}, X_p^{r}, X_{p+\lambda}^{t}, X_{p+\lambda+1}^{j}, \ldots, X_q^{K},$$

where

$$n_1+n_2+n_3+\ldots+n_{p-2}+n_{p-1}+S=n-1,$$

and

$$m_1+m_2+m_3+\ldots+m_{p-2}+m_{p-1}+r+t+j+\ldots+K=n.$$

Assume that (γ) is parametric and (δ) principal. In the case where at least one of the m's is greater than the n with the same subscript, it is obvious that no derivative of (γ) with respect to $x_p, x_{p+1}, \ldots, x_q$ is identical with (δ) or a derivative of (δ). But, suppose,

$$n_i \geqq m_i, \qquad (i=1, 2, \ldots, p-1).$$

There are two cases to be disposed of. Firstly, let $r>S$. Since (δ) is principal,

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}+1}, X_p^{f}, X_{p+\lambda}^{t}, X_{p+\lambda+1}^{j}, \ldots, X_q^{K}$$

is surely principal, and it follows from the property K that

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}+1}, X_p^{f}, X_{p+\lambda}^{t}, X_{p+\lambda+1}^{j}, \ldots, X_q^{K-1}$$

is principal. And a repeated application of the above process shows us that

$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-2}^{n_{p-2}}, X_{p-1}^{n_{p-1}}, X_p^{s+1}$$

is certainly principal. Hence x_p can not occur actually in the arbitrary function ϕ which corresponds to (γ). And since $r>S$ no derivative of (γ) with respect to $x_{p+1}, x_{p+2}, \ldots, x_q$ is identical with (δ) or with a derivative of (δ).

Now let us consider the case where $r \leqq S$. Then

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}}, X_p^r, X_{p+1}, X_{p+\lambda}^t, \ldots, X_q^{K-1}$$

is principal and continuing the process, we show that

$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-2}^{n_{p-2}}, X_{p-1}^{n_{p-1}}, X_p^s, X_{p+1}$$

is principal. In a similar fashion we can show that

$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-2}^{n_{p-2}}, X_{p-1}^{n_{p-1}}, X_p^s, X_{p+2},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-2}^{n_{p-2}}, X_{p-1}^{n_{p-1}}, X_p^s, X_{p+\lambda},$$

are principal, and we write our initial condition

$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^s = \phi(x_{p+\lambda+1}, \ldots).$$

Comparing this with

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_q^K = M,$$

and prolonging the two equations, we shall arrive at no incompatibility since the second expression contains a factor $X_{p+\lambda}^f$ where $f \neq 0$. Thus we have given a negative response to the first question.

Solution of the Second Question.

Let us write

$$\left.\begin{array}{ll} X_1^{n_1}, X_2^{n_2}, \ldots, X_p^{n_p} = \phi(x_p, x_{p+1}, \ldots), & n_p \neq 0, \\ X_1^{m_1}, X_2^{m_2}, \ldots, X_p^{m_p} = \psi(x_p, x_{p+1}, \ldots), & m_p \neq 0, \end{array}\right\} \qquad \text{(III)}$$

where

$$n_1 + n_2 + \ldots + n_p = n-1,$$
$$m_1 + m_1 + \ldots + m_p = n-1.$$

At least one of the m's differs from the corresponding n, for the two expressions are assumed to be different. Then since the sum of the m's equals the sum of the n's for one value of i at least, $m_i > n_i$, and for at least one value of j, which letter is different from i, we have the inequality $n_j > m_j$. It follows that the exponents of at least two of the X's in each of the expressions of (III) are different. Therefore, we have the inequality $n_i \neq m_i$ where $i \neq p$. Therefore, if we differentiate (III) an arbitrary number of times with respect to x_p, $x_{p+1}, \ldots, x_q$, we shall never arrive at incompatibilities. In case $m_p = 0$, $n_p \neq 0$, the course of reasoning is not different, for we write

$$X_1^{n_1}, X_2^{n_2}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{n_p} = \phi(x_p, x_{p+1}, \ldots),$$
$$X_1^{m_1}, X_2^{m_2}, \ldots, X_{p-1}^{m_{p-1}} = \psi(x_{p-1}, x_p, \ldots).$$

Then at least one of the m's is greater than the corresponding n, and again we see that no incompatibility can arise.

All that we have now proven is only a generalization of section 197 of Riquier's book.

Section 3.

The facts which are now at our disposal will be used to rectify a defect in the work of Delassus. This author, in the paper we have cited,* has reduced any set of equations to a canonical form of which, however, there are exceptional cases. The determinant

$$\Delta=\frac{\partial(\xi_1, \xi_2, \ldots, \xi_{p-1}, \zeta_1)}{\partial(\xi_1', \xi_2', \ldots, \xi_p')},$$

mentioned on page 436 of his work actually does vanish for certain exceptional cases. The following example makes this clear. Indeed, the reasoning of the last paragraphs of page 436, and the first paragraphs of the following page is a little defective.

It is easy to show that if we are given a set of partial differential equations, there is no unique canonical form which is at the same time orthonomic.

To prove this let us take a system of partial differential equations of the second order containing three equations. When there are only one unknown and three independent variables, the maximum number of derivatives which may be involved in this system is six. Denote these by X_1^2, X_1X_2, X_1X_3, X_2^2, X_2X_3, X_3^2, and take them three at a time. There will be twenty different combinations.

Let us suppose that X_1^2, X_1X_2, X_1X_3 are the first members of our system. In other words let us write the system thus:

$$X_1^2=f_1, \quad X_1X_2=f_2, \quad X_1X_3=f_3, \tag{I}$$

the second members being functions of the independent variables x_1, x_2 and x_3 only. Let us now consider the system

$$X_1^2=\phi_1, \quad X_1X_2=\phi_2, \quad X_2^2=\phi_3, \tag{II}$$

the second members being of the same type as those of the first system. Furthermore, let them satisfy the conditions for the passivity of system (II). We can easily show that no linear transformation of variables will send the set (II) into a set of type (I). For, make the change,

$$\begin{aligned} x_1'&=ax_1+bx_2+cx_3, \\ x_2'&=a'x_1+b'x_2+c'x_3, \\ x_3'&=a''x_1+b''x_2+c''x_3. \end{aligned}$$

* *Annales de l'école normale supérieure,* Vol. IX (1896).

Then,

$$X_1^2=a^2X_1'^2 +2aa'X_1'X_2'+2aa''X_1'X_3'+\dots\dots,$$
$$X_1X_2=abX_1'^2+(ab'+a'b)X_1'X_2'+(ab''+a''b)X_1'X_3'+\dots\dots,$$
$$X_2^2= b^2X_1^2 +2bb'X_1'X_2'+2bb''X_1'X_3'+\dots\dots,$$

and it is easy to see that the determinant

$$\begin{vmatrix} a^2, & 2aa', & 2aa'', \\ ab, & ab'+a'b, & ab''+a''b, \\ b^2, & 2bb', & 2bb'', \end{vmatrix}$$

vanishes.

We next consider the possibility of transforming (II) into a system whose first members are X_1^2, X_1X_2, X_2X_3. We easily see that no choice of 'cotes' will make these three quantities normal to all three of the expressions X_1X_3, X_2^2, X_3^2, and therefore a change of variables followed by a resolution, leads to a set which is non-orthonomic. And if we examine the eighteen cases which remain, we shall see that we shall obtain similar results each time.

The above simple examples show us that cases exist where it is impossible to make a reduction to a fixed canonical form which is also orthonomic.

Section 4.

In what follows we shall show that it is always possible, by the aid of a linear homogeneous transformation of the independent variables, to put any system of equations into a form which shall be a special case of what we have called the canonical form. This form, indeed, varies with the system of equations, but is so defined that when the original system is given, we can say what the canonical form shall be. Likewise we shall see that we can determine whether or not the system is passive by a finite number of operations, since our canonical form is regular. From now on we shall call the canonical form, defined in the first section, form (A), and the new form about to be defined, form (B). (A) contains (B), that is if a system enters into form (B), it certainly belongs also to (A).

We shall, to simplify writing, define our new form in the case of three independent variables only. It is characterized by the following property:

If $X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}$ *is a first member of the set, then*

$$X_1^{\alpha_1}X_2^{\alpha_2+1}X_3^{\alpha_3-1},\quad X_1^{\alpha_1+1}X_2^{\alpha_2}X_3^{\alpha_3-1},\quad X_1^{\alpha_1+1}X_2^{\alpha_2-1}X_3^{\alpha_3},$$

are also first members. In fact any quantity is a first member which possesses the double property of

(a) being obtained from $X_1^{a_1}X_2^{a_2}X_3^{a_3}$ *by adding unity to one exponent, and subtracting it from another.*

(b) being anterior to $X_1^{a_1}X_2^{a_2}X_3^{a_3}$ *according to the definition of anterior given by Delassus.*

Let us call this double property (J). It is easy to see that this new form is less general than the canonical form (A). For if a system possesses the property (K), the existence of $X_1^{a_1}X_2^{a_2}X_3^{a_3}$ among the group of first members involves the existence of $X_1^{a_1}X_2^{a_2+1}X_3^{a_3-1}$ and $X_1^{a_1+1}X_2^{a_2}X_3^{a_3-1}$ in the same group, but not the existence of $X_1^{a_1+1}X_2^{a_2-1}X_3^{a_3}$.

In the second section we have shown that if a system possesses the property (K), the prolonged system possesses the same property. By the same method we can show that if a system possesses the property (J) the prolonged system possesses it also.

Section 5.

We shall now proceed to develop a method by which we can transform the most general system of partial differential equations into canonical form (B). Firstly, we shall demonstrate the following lemma:

Given an equation

$$\delta = f(x, y, \ldots, \sigma, \ldots, \tau, \ldots),$$

belonging to the most general system S, where τ and σ represent the derivatives of a class greater and less than δ, respectively. Suppose that it is desirable to solve the system according to the general method employed for the solution of implicit functions, and that the initial values are $x_0, y_0, \ldots, \sigma_0, \ldots, \tau_0, \ldots, \delta_0$. It is always possible to solve the system S in such a way that the initial values of all derivatives of the type $\frac{\partial f}{\partial \tau}$ shall vanish.*

For, consider the system

$$\delta_1 = f_1(x, y, \ldots, \sigma, \ldots, \tau, \ldots),\ \delta_2 = f_2,\ \ldots,\ \delta_n = f_n.$$

Suppose that $\frac{\partial f_1}{\partial \tau_1} \neq 0$. Then we can solve the first equation of the system with respect to τ_1 and eliminate τ_1 from the second members of the remaining equations of the set.† Since τ_1 does not occur among the first members of these remaining equations, nor $\delta_2, \delta_3, \ldots, \delta_n$ among the arguments of the first

* For the meaning of this word see Riquier, *loc. cit.*, p. 208.

† τ_1 is any one of the τ's.

equation, or of the first equation solved with respect to τ_1 the system can now be written:

$$\tau_1=\bar{f}_1(xy,\ \ldots,\ \delta_1,\ \sigma,\ \ldots,\ \tau,\ \ldots),\ \delta_2=\bar{f}_2,\ \delta_3=\bar{f}_3,\ \ldots,\ \delta_n=f_n.$$

And so we can repeat this process until we exhaust all possibilities and obtain a system of the type desired.

We will now suppose, in order to fix ideas, that our system is composed of equations all of order n, and involving only one unknown u. Furthermore, let it be of the type described in the lemma. Finally we shall write it

$$S\left\{\begin{array}{l} \xi_1 \quad +\phi_1 \quad (\eta\ldots.)=0, \\ \xi_2 \quad +\phi_2 \quad (\eta\ldots.)=0, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ \xi_{p-1}+\phi_{p-1}(\eta\ldots.)=0, \\ \zeta \quad +\phi_p \quad (\eta\ldots.)=0. \end{array}\right.$$

If we suppose all the derivatives of order n written down in the order adopted by Delassus,* ξ_2 shall be supposed posterior to ξ_1, ξ_3 posterior to ξ_2, etc. But we do not assert that ξ_2 follows immediately after ξ_1, or that ξ_3 follows immediately after ξ_2, etc. There may be intermediate derivatives. As for ζ we make no hypotheses concerning it. It may occur anywhere in the range.

We have assumed our system to be of the type described in the lemma. This means that if in the equation

$$\xi_i-\phi_i=0,$$

ϕ_i contains some derivative, say η anterior to ξ_i, then $\dfrac{\partial\phi_i}{\partial\eta}=0$ at the initial points $x_0,\ y_0,\ \ldots,\ \sigma_0,\ \ldots,\ \tau_0,\ \ldots$, of the solution.

Now let

$$\zeta=X_1^{a_1}X_2^{a_2}X_3^{a_3}.$$

Suppose $X_1^{a_1}X_2^{a_2}X_3^{a_3}$ to be absent from the group of first members of our system. We shall call it ζ_p. We are going to demonstrate that the system whose first members are

$$\xi_1,\ \xi_2,\ \ldots,\ \xi_{p-1},\ \zeta,$$

can, after a linear homogeneous transformation of the independent variables,

$$\begin{aligned} x_1'&=ax_1+bx_2+cx_3, \\ x_2'&=a'x_1+b'x_2+c'x_3, \\ x_3'&=a''x_1+b''x_2+c''x_3, \end{aligned}$$

* *Annales de l'école normale supérieure, loc. cit.*, p. 426.

followed by a resolution with respect to the proper derivatives, be changed into a system whose first members are

$$\xi'_1, \xi'_2, \ldots, \xi'_{p-1}, \xi'_p.$$

It will be sufficient to indicate the points wherein our proof differs from that of Delassus.

Let us return to the system S. It is our object to solve this system with respect to $\xi'_1, \xi'_2, \ldots, \xi'_p$, starting from the initial values $\xi_1^{(0)}, \xi_2^{(0)}, \ldots, \xi_{(p-1)}^{(0)}$, $\zeta^0, \eta^0, \ldots$, and we must show that the functional determinant does not vanish for these initial values. The functional determinant of our p equations, after they have been transformed with respect to $\xi'_1, \xi'_2, \ldots, \xi'_p$ is,

$$D=\begin{vmatrix} \dfrac{\partial\xi_1}{\partial\xi'_1}+\Sigma\dfrac{\partial\phi_1}{\partial\eta}\dfrac{\partial\eta}{\partial\xi'_1}, \ldots, \dfrac{\partial\xi_1}{\partial\xi'_p}+\Sigma\dfrac{\partial\phi_1}{\partial\eta}\dfrac{\partial\eta}{\partial\xi'_p}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \dfrac{\partial\xi_{p-1}}{\partial\xi'_1}+\Sigma\dfrac{\partial\phi_{p-1}}{\partial\eta}\dfrac{\partial\eta}{\partial\xi'_1}, \ldots, \dfrac{\partial\xi_{p-1}}{\partial\xi'_p}+\Sigma\dfrac{\partial\phi_{p-1}}{\partial\eta}\dfrac{\partial\eta}{\partial\xi'_p}, \\ \dfrac{\partial\zeta}{\partial\xi'_1}+\Sigma\dfrac{\partial\phi_p}{\partial\eta}\dfrac{\partial\eta}{\partial\xi'_1}, \ldots, \dfrac{\partial\zeta}{\partial\xi'_p}+\Sigma\dfrac{\partial\phi_p}{\partial\eta}\dfrac{\partial\eta}{\partial\xi'_p}. \end{vmatrix}$$

If $\dfrac{\partial\phi_\lambda}{\partial\eta}\neq 0$ for the initial values, then by hypothesis $\dfrac{\partial\xi_\lambda}{\partial\xi'_\lambda}$ is anterior to η, $\{\lambda=1, 2, \ldots, p\}$.

And so following the same course of reasoning as Delassus, we retain only the first members of each element of the determinant. We are left with

$$\Delta=\frac{\partial(\xi_1, \xi_2, \ldots, \xi_{p-1}, \zeta)}{\partial(\xi'_1, \xi'_2, \ldots, \xi'_p)}.$$

Then consider the minors of the above determinant $A_{p,p}, A_{p,p-1}, \ldots, A_{p,1}$. The diagonal of $A_{p,p}$ consists of the elements

$$\frac{\partial\xi_1}{\partial\xi'_1}, \frac{\partial\xi_2}{\partial\xi'_2}, \ldots, \frac{\partial\xi_{p-1}}{\partial\xi'_{p-1}}.$$

Each of these terms is of degree n in a, b', c''. For let $\xi'_k=X_1'^{a_1}X_2'^{a_2}X_3'^{a_3}$. Then, clearly, $\xi_k=X_1^{a_1}X_2^{a_2}X_3^{a_3}$, where $\{k=1, 2, \ldots, p-1\}$. After transforming we obtain

$$X_1^{a_1}X_2^{a_2}X_3^{a_3}=(aX'_1+a'X'_2+a''X'_3)^{a_1}(bX'_1+b'X'_2+b''X'_3)^{a_2}(cX'_1+c'X'_2+c''X'_3)^{a_3}.$$

The coefficient of $X_1'^{a_1}X_2'^{a_2}X_3'^{a_3}$ is $a^{a_1}b'^{a_2}c''^{a_3}$. Hence $\dfrac{\partial\xi_k}{\partial\xi'_k}=a^{a_1}b'^{a_2}c''^{a_3}$, which contains no other quantities except a, b', c'', and is, consequently, of the n-th degree

in the above-mentioned quantities. So $A_{pp} \not\equiv 0$, but is of degree $n(p-1)$ in a, b', c''.

Let us now consider the expression $\dfrac{\partial \zeta_1}{\partial \xi'_p}$. Let $\zeta_1 = X_1^{a_1} X_2^{a_2} X_3^{a_3}$. Then,

$$\xi'_p = X_1^{a_1} X_2^{a_2-1} X_3^{a_3+1}, \quad \text{or} \quad X_1^{a_1-1} X_2^{a_2+1} X_3^{a_3}, \quad \text{or} \quad X_1^{a_1-1} X_2^{a_2} X_3^{a_3+1}.$$

Let us suppose that the first equality is the one which exists, for the reasoning will be the same in the other cases:

$$X_1^{a_1} X_2^{a_2} X_3^{a_3} = (aX'_1 + a'X'_2 + a''X'_3)^{a_1} (bX'_1 + b'X_2 + b''X_3)^{a_2} (cX'_1 + c'X'_2 + c''X'_3)^{a_3}.$$

The coefficient of $X_1^{a_1} X_2^{a_2-1} X_3^{a_3+1}$ is clearly obtained by expanding

$$a_1^{a_1} b'^{a_2-1} c''^{a_3} X_1'^{a_1} X_2^{a_2-1} X_3^{a_3} (bX'_1 + b'X_2 + b''X_3),$$

and is clearly $a^{a_1} b'^{a_2-1} b'' c''^{a_3}$ and is of degree $n-1$ in a, b', c''. Hence $\dfrac{\partial \zeta_1}{\partial \xi'_p}$ is of degree $n-1$ in the same quantities.

We can easily convince ourselves that $\dfrac{\partial \zeta_1}{\partial \xi'_1}, \dfrac{\partial \zeta_1}{\partial \xi'_2}, \ldots, \dfrac{\partial \zeta_1}{\partial \xi'_{p-1}}$ are of degree $\not> (n-1)$ in the a, b', c''. Also we easily demonstrate that all the minors $A_{p,\,p-1}, \ldots, A_{p,1}$ are of degree less than $n(p-1)$. Therefore, the whole determinant does not vanish identically, but is effectively of degree $n(p-1) + (n-1)$ in a, b', c''.

Thus we see that if $X_1^{a_1} X_2^{a_2} X_3^{a_3}$ is among the group of first members, either $X_1^{a_1} X_2^{a_2+1} X_3^{a_3}$ is a first member, or it can be made a first member by a linear homogeneous transformation followed by a resolution. Similar reasoning will show that we can make the same statement with respect to $X_1^{a_1+1} X_2^{a_2} X_3^{a_3-1}$ and $X_1^{a_1+1} X_2^{a_2-1} X_3^{a_3}$.

We can follow a course of reasoning parallel to that of Delassus in case the various equations of the system are not all of order n, or if there be more unknowns than one. It is easy to see that we can establish the convergence of the series formally satisfying our canonical system of equations by the method of Delassus, *i. e.*, by reducing the given system to a system of systems of the type of Kovalefski.

We should also note that a system can often be made canonical in more than one way. For example:

$$\frac{\partial^2 u}{\partial x_1^2} = f_1(x_1 x_2 x_3), \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} = f_2 \; \frac{\partial^2 u}{\partial x_1 \partial x_3} = c \frac{\partial^2 u}{\partial x_2^2},$$

and

$$\frac{\partial^2 u}{\partial x_1^2} = f_1(x_1 x_2 x_3), \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} = f_2 \; \frac{\partial^2 u}{\partial x_2} = \frac{1}{c} \frac{\partial^2 u}{\partial x_1 \partial x_3},$$

are equivalent forms, and both are canonical.

So far the author has shown, that since the canonical form developed is regular, it can be determined whether or not it is passive by a finite number of operations. But the knowledge that the canonical form is regular is not necessary for this purpose. In fact, it can readily be shown, that using the known theorems concerning the invariants λ and μ, we can determine whether or not a given system at certain points not solvable in orthonomic form, is passive throughout the entire space of n dimensions.